

# WELL-POSEDNESS AND SCATTERING FOR FOURTH ORDER NONLINEAR SCHRÖDINGER TYPE EQUATIONS AT THE SCALING CRITICAL REGULARITY

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**ABSTRACT.** In the present paper, we consider the Cauchy problem of fourth order nonlinear Schrödinger type equations with a derivative nonlinearity. In one dimensional case, we prove that the fourth order nonlinear Schrödinger equation with the derivative quartic nonlinearity  $\partial_x(\overline{u}^4)$  is the small data global in time well-posed and scattering to a free solution. Furthermore, we show that the same result holds for the  $d \geq 2$  and derivative polynomial type nonlinearity, for example  $|\nabla|(u^m)$  with  $(m-1)d \geq 4$ .

*Key Words and Phrases.* Schrödinger equation, well-posedness, Cauchy problem, scaling critical, multilinear estimate, bounded  $p$ -variation.

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## 1. INTRODUCTION

We consider the Cauchy problem of the fourth order nonlinear Schrödinger type equations:

$$\begin{cases} (i\partial_t + \Delta^2)u = \partial P_m(u, \overline{u}), & (t, x) \in (0, \infty) \times \mathbb{R}^d \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d \end{cases} \quad (1.1)$$

where  $m \in \mathbb{N}$ ,  $m \geq 2$ ,  $P_m$  is a polynomial which is written by

$$P_m(f, g) = \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0} \\ \alpha + \beta = m}} f^\alpha g^\beta,$$

$\partial$  is a first order derivative with respect to the spatial variable, for example a linear combination of  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}$  or  $|\nabla| = \mathcal{F}^{-1}[|\xi|\mathcal{F}]$  and the unknown function  $u$  is  $\mathbb{C}$ -valued. The fourth order Schrödinger equation with  $P_m(u, \overline{u}) = |u|^{m-1}u$  appears in the study of deep water wave dynamics [2], solitary waves [14], [15], vortex filaments [3], and so on. The equation (1.1) is invariant under the following scaling transformation:

$$u_\lambda(t, x) = \lambda^{-3/(m-1)} u(\lambda^{-4}t, \lambda^{-1}x),$$

and the scaling critical regularity is  $s_c = d/2 - 3/(m - 1)$ . The aim of this paper is to prove the well-posedness and the scattering for the solution of (1.1) in the scaling critical Sobolev space.

There are many results for the fourth order nonlinear Schrödinger equation with derivative nonlinearities (see [17], [18], [11], [6], [7], [12], [19], [13], [20], [8], [9], and references cited therein). Especially, the one dimensional case is well studied. Wang ([20]) considered (1.1) for the case  $d = 1$ ,  $m = 2l + 1$ ,  $l \geq 2$ ,  $P_{2l+1}(u, \bar{u}) = |u|^{2l}u$  and proved the small data global in time well-posedness for  $s = s_c$  by using Kato type smoothing effect. But he did not treat the cubic case. Actually, a technical difficulty appears in this case (see Theorem 1.8 below).

Hayashi and Naumkin ([8]) considered (1.1) for  $d = 1$  with the power type nonlinearity  $\partial_x(|u|^{\rho-1}u)$  ( $\rho > 4$ ) and proved the global existence of the solution and the scattering in the weighted Sobolev space. Moreover, they ([9]) also proved that the large time asymptotics is determined by the self similar solution in the case  $\rho = 4$ . Therefore, derivative quartic nonlinearity in the one spatial dimension is the critical in the sense of the asymptotic behavior of the solution.

We firstly focus on the quartic nonlinearity  $\partial_x(\bar{u}^4)$  in one space dimension. Since this nonlinearity has some good structure, the global solution scatters to a free solution in the scaling critical Sobolev space. Our argument does not apply to (1.1) with  $P(u, \bar{u}) = |u|^3u$  because we rely on the Fourier restriction norm method. Now, we give the first results in this paper. For a Banach space  $H$  and  $r > 0$ , we define  $B_r(H) := \{f \in H \mid \|f\|_H \leq r\}$ .

**Theorem 1.1.** *Let  $d = 1$ ,  $m = 4$  and  $P_4(u, \bar{u}) = \bar{u}^4$ . Then the equation (1.1) is globally well-posed for small data in  $\dot{H}^{-1/2}$ . More precisely, there exists  $r > 0$  such that for any  $T > 0$  and all initial data  $u_0 \in B_r(\dot{H}^{-1/2})$ , there exists a solution*

$$u \in \dot{Z}_r^{-1/2}([0, T)) \subset C([0, T); \dot{H}^{-1/2})$$

*of (1.1) on  $(0, T)$ . Such solution is unique in  $\dot{Z}_r^{-1/2}([0, T))$  which is a closed subset of  $\dot{Z}^{-1/2}([0, T))$  (see Definition 2.11 and (4.2)). Moreover, the flow map*

$$S_T^+ : B_r(\dot{H}^{-1/2}) \ni u_0 \mapsto u \in \dot{Z}^{-1/2}([0, T))$$

*is Lipschitz continuous.*

**Remark 1.2.** *We note that  $s = -1/2$  is the scaling critical exponent of (1.1) for  $d = 1$ ,  $m = 4$ .*

**Corollary 1.3.** *Let  $r > 0$  be as in Theorem 1.1. For all  $u_0 \in B_r(\dot{H}^{-1/2})$ , there exists a solution  $u \in C([0, \infty); \dot{H}^{s_c})$  of (1.1) on  $(0, \infty)$  and the solution scatters in  $\dot{H}^{-1/2}$ . More precisely, there exists  $u^+ \in \dot{H}^{-1/2}$  such that*

$$u(t) - e^{it\Delta^2} u^+ \rightarrow 0 \text{ in } \dot{H}^{-1/2} \text{ as } t \rightarrow +\infty.$$

Moreover, we obtain the large data local in time well-posedness in the scaling critical Sobolev space. To state the result, we put

$$B_{\delta,R}(H^s) := \{u_0 \in H^s \mid u_0 = v_0 + w_0, \|v_0\|_{\dot{H}^{-1/2}} < \delta, \|w_0\|_{L^2} < R\}$$

for  $s < 0$ .

**Theorem 1.4.** *Let  $d = 1$ ,  $m = 4$  and  $P_4(u, \bar{u}) = \bar{u}^4$ . Then the equation (1.1) is locally in time well-posed in  $H^{-1/2}$ . More precisely, there exists  $\delta > 0$  such that for all  $R \geq \delta$  and  $u_0 \in B_{\delta,R}(H^{-1/2})$  there exists a solution*

$$u \in Z^{-1/2}([0, T]) \subset C([0, T]; H^{-1/2})$$

for  $T = \delta^8 R^{-8}$  of (1.1).

Furthermore, the same statement remains valid if we replace  $H^{-1/2}$  by  $\dot{H}^{-1/2}$  as well as  $Z^{-1/2}([0, T])$  by  $\dot{Z}^{-1/2}([0, T])$ .

**Remark 1.5.** *For  $s > -1/2$ , the local in time well-posedness in  $H^s$  follows from the usual Fourier restriction norm method, which covers for all initial data in  $H^s$ . It however is not of very much interest. On the other hand, since we focus on the scaling critical cases, which is the negative regularity, we have to impose that the  $\dot{H}^{-1/2}$  part of initial data is small. But, Theorem 1.4 is a large data result because the  $L^2$  part is not restricted.*

The main tools of the proof are the  $U^p$  space and  $V^p$  space which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([4], [5]).

We also consider the one dimensional cubic case and the high dimensional cases. The second result in this paper is as follows.

**Theorem 1.6.** (i) *Let  $d = 1$  and  $m = 3$ . Then the equation (1.1) is locally well-posed in  $H^s$  for  $s \geq 0$ .*

(ii) *Let  $d \geq 2$  and  $(m-1)d \geq 4$ . Then the equation (1.1) is globally well-posed for small data in  $\dot{H}^{s_c}$  (or  $H^s$  for  $s \geq s_c$ ) and the solution scatters in  $\dot{H}^{s_c}$  (or  $H^s$  for  $s \geq s_c$ ).*

The smoothing effect of the linear part recovers derivative in higher dimensional case. Therefore, we do not use the  $U^p$  and  $V^p$  type spaces. More precisely, to establish Theorem 1.6, we only use the Strichartz estimates and get the solution in  $C([0, T]; H^{s_c}) \cap L^{p_m}([0, T]; W^{q_m, s_c+1/(m-1)})$  with  $p_m = 2(m-1)$ ,  $q_m = 2(m-1)d/\{(m-1)d-2\}$ . Accordingly, the scattering follows from a standard argument. Since the condition  $(m-1)d \geq 4$  is equivalent to  $s_c + 1/(m-1) \geq 0$ , the solution space  $L^{p_m}([0, T]; W^{q_m, s_c+1/(m-1)})$  has nonnegative regularity even if the data belongs to  $H^{s_c}$  with  $-1/(m-1) \leq s_c < 0$ . Our proof of Theorem 1.6 (ii) cannot be applied for  $d = 1$  since the Schrödinger admissible  $(a, b)$  in (5.3) does not exist.

**Remark 1.7.** *For the case  $d = 1$ ,  $m = 4$  and  $P_4(u, \bar{u}) \neq \bar{u}^4$ , we can obtain the local in time well-posedness of (1.1) in  $H^s$  for  $s \geq 0$  by the same way of the proof of Theorem 1.6. Actually, we can get the solution in  $C([0, T]; H^s) \cap L^4([0, T]; W^{s+1/2, \infty})$  for  $s \geq 0$  by using the iteration argument since the fractional Leibnitz rule (see [1]) and the Hölder inequality imply*

$$\left\| |\nabla|^{s+\frac{1}{2}} \prod_{j=1}^4 u_j \right\|_{L_t^{4/3}([0, T]; L_x^1)} \lesssim T^{1/4} \left\| |\nabla|^{s+\frac{1}{2}} u_1 \right\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^4 L_x^\infty} \|u_3\|_{L_t^\infty L_x^2} \|u_4\|_{L_t^\infty L_x^2}.$$

We give a remark on our problem, which shows that the standard iteration argument does not work.

**Theorem 1.8.** (i) *Let  $d = 1$ ,  $m = 3$ ,  $s < 0$  and  $P_3(u, \bar{u}) = |u|^2 u$ . Then the flow map of (1.1) from  $H^s$  to  $C(\mathbb{R}; H^s)$  is not smooth.*

(ii) *Let  $m \geq 2$ ,  $s < s_c$  and  $\partial = |\nabla|$  or  $\frac{\partial}{\partial x_k}$  for some  $1 \leq k \leq d$ . Then the flow map of (1.1) from  $H^s$  to  $C(\mathbb{R}; H^s)$  is not smooth.*

More precisely, we prove that the flow map is not  $C^3$  if  $d = 1$ ,  $m = 3$ ,  $s < 0$  and  $P_3(u, \bar{u}) = |u|^2 u$  or  $C^m$  if  $d \geq 1$ ,  $m \geq 2$ , and  $s < s_c$ . It leads that the standard iteration argument fails, because the flow map is smooth if it works. Of course, there is a gap between ill-posedness and absence of a smooth flow map.

Since the resonance appears in the case  $d = 1$ ,  $m = 3$  and  $P_3(u, \bar{u}) = |u|^2 u$ , there exists an irregular flow map even for the subcritical Sobolev regularity.

**Notation.** We denote the spatial Fourier transform by  $\widehat{\cdot}$  or  $\mathcal{F}_x$ , the Fourier transform in time by  $\mathcal{F}_t$  and the Fourier transform in all variables by  $\widetilde{\cdot}$  or  $\mathcal{F}_{tx}$ . The free evolution  $S(t) := e^{it\Delta^2}$  is given as a Fourier multiplier

$$\mathcal{F}_x[S(t)f](\xi) = e^{-it|\xi|^4} \widehat{f}(\xi).$$

We will use  $A \lesssim B$  to denote an estimate of the form  $A \leq CB$  for some constant  $C$  and write  $A \sim B$  to mean  $A \lesssim B$  and  $B \lesssim A$ . We will use the convention that capital letters denote dyadic numbers, e.g.  $N = 2^n$  for  $n \in \mathbb{Z}$  and for a dyadic summation we write  $\sum_N a_N := \sum_{n \in \mathbb{Z}} a_{2^n}$  and  $\sum_{N \geq M} a_N := \sum_{n \in \mathbb{Z}, 2^n \geq M} a_{2^n}$  for brevity. Let  $\chi \in C_0^\infty((-2, 2))$  be an even, non-negative function such that  $\chi(t) = 1$  for  $|t| \leq 1$ . We define  $\psi(t) := \chi(t) - \chi(2t)$  and  $\psi_N(t) := \psi(N^{-1}t)$ . Then,  $\sum_N \psi_N(t) = 1$  whenever  $t \neq 0$ . We define frequency and modulation projections

$$\widehat{P_N u}(\xi) := \psi_N(\xi) \widehat{u}(\xi), \quad \widetilde{Q_M^S u}(\tau, \xi) := \psi_M(\tau - |\xi|^4) \widetilde{u}(\tau, \xi).$$

Furthermore, we define  $Q_{\geq M}^S := \sum_{N \geq M} Q_N^S$  and  $Q_{< M}^S := Id - Q_{\geq M}^S$ .

The rest of this paper is planned as follows. In Section 2, we will give the definition and properties of the  $U^p$  space and  $V^p$  space. In Section 3, we will give the multilinear estimates which are main estimates to prove Theorems 1.1 and 1.4. In Section 4, we will give the proof of the well-posedness and the scattering (Theorem 1.1, Corollary 1.3, and Theorem 1.4). In Section 5, we will give the proof of Theorem 1.6. In Section 6, we will give the proof of Theorem 1.8.

## 2. THE $U^p$ , $V^p$ SPACES AND THEIR PROPERTIES

In this section, we define the  $U^p$  space and the  $V^p$  space, and introduce the properties of these spaces which are proved by Hadac, Herr and Koch ([4], [5]).

We define the set of finite partitions  $\mathcal{Z}$  as

$$\mathcal{Z} := \{ \{t_k\}_{k=0}^K \mid K \in \mathbb{N}, -\infty < t_0 < t_1 < \cdots < t_K \leq \infty \}$$

and if  $t_K = \infty$ , we put  $v(t_K) := 0$  for all functions  $v : \mathbb{R} \rightarrow L^2$ .

**Definition 2.1.** Let  $1 \leq p < \infty$ . For  $\{t_k\}_{k=0}^K \in \mathcal{Z}$  and  $\{\phi_k\}_{k=0}^{K-1} \subset L^2$  with  $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$  we call the function  $a : \mathbb{R} \rightarrow L^2$  given by

$$a(t) = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)}(t) \phi_{k-1}$$

a “ $U^p$ -atom”. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \mid a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \mid u = \sum_{j=1}^{\infty} \lambda_j a_j, a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \right\}.$$

**Definition 2.2.** Let  $1 \leq p < \infty$ . We define the space of the bounded  $p$ -variation

$$V^p := \{v : \mathbb{R} \rightarrow L^2 \mid \|v\|_{V^p} < \infty\}$$

with the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}.$$

Likewise, let  $V_{-,rc}^p$  denote the closed subspace of all right-continuous functions  $v \in V^p$  with  $\lim_{t \rightarrow -\infty} v(t) = 0$ , endowed with the same norm  $\|\cdot\|_{V^p}$ .

**Proposition 2.3** ([4] Proposition 2.2, 2.4, Corollary 2.6). Let  $1 \leq p < q < \infty$ .

- (i)  $U^p$ ,  $V^p$  and  $V_{-,rc}^p$  are Banach spaces.
- (ii) For every  $v \in V^p$ ,  $\lim_{t \rightarrow -\infty} v(t)$  and  $\lim_{t \rightarrow \infty} v(t)$  exist in  $L^2$ .
- (iii) The embeddings  $U^p \hookrightarrow V_{-,rc}^p \hookrightarrow U^q \hookrightarrow L_t^\infty(\mathbb{R}; L_x^2(\mathbb{R}^d))$  are continuous.

**Theorem 2.4** ([4] Proposition 2.10, Remark 2.12). Let  $1 < p < \infty$  and  $1/p + 1/p' = 1$ . If  $u \in V_{-,rc}^1$  be absolutely continuous on every compact intervals, then

$$\|u\|_{U^p} = \sup_{v \in V^{p'}, \|v\|_{V^{p'}} = 1} \left| \int_{-\infty}^{\infty} (u'(t), v(t))_{L^2(\mathbb{R}^d)} dt \right|.$$

**Definition 2.5.** Let  $1 \leq p < \infty$ . We define

$$U_S^p := \{u : \mathbb{R} \rightarrow L^2 \mid S(-\cdot)u \in U^p\}$$

with the norm  $\|u\|_{U_S^p} := \|S(-\cdot)u\|_{U^p}$ ,

$$V_S^p := \{v : \mathbb{R} \rightarrow L^2 \mid S(-\cdot)v \in V_{-,rc}^p\}$$

with the norm  $\|v\|_{V_S^p} := \|S(-\cdot)v\|_{V^p}$ .

**Remark 2.6.** The embeddings  $U_S^p \hookrightarrow V_S^p \hookrightarrow U_S^q \hookrightarrow L^\infty(\mathbb{R}; L^2)$  hold for  $1 \leq p < q < \infty$  by Proposition 2.3.

**Proposition 2.7** ([4] Corollary 2.18). Let  $1 < p < \infty$ . We have

$$\|Q_{\geq M}^S u\|_{L_{tx}^2} \lesssim M^{-1/2} \|u\|_{V_S^2}, \quad (2.1)$$

$$\|Q_{< M}^S u\|_{V_S^p} \lesssim \|u\|_{V_S^p}, \quad \|Q_{\geq M}^S u\|_{V_S^p} \lesssim \|u\|_{V_S^p}, \quad (2.2)$$

**Proposition 2.8** ([4] Proposition 2.19). Let

$$T_0 : L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \rightarrow L_{loc}^1(\mathbb{R}^d)$$

be a  $m$ -linear operator. Assume that for some  $1 \leq p, q < \infty$

$$\|T_0(S(\cdot)\phi_1, \dots, S(\cdot)\phi_m)\|_{L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^d))} \lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{R}^d)}.$$

Then, there exists  $T : U_S^p \times \dots \times U_S^p \rightarrow L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^d))$  satisfying

$$\|T(u_1, \dots, u_m)\|_{L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^d))} \lesssim \prod_{i=1}^m \|u_i\|_{U_S^p}$$

such that  $T(u_1, \dots, u_m)(t)(x) = T_0(u_1(t), \dots, u_m(t))(x)$  a.e.

Now we refer the Strichartz estimate for the fourth order Schrödinger equation proved by Pausader. We say that a pair  $(p, q)$  is admissible if  $2 \leq p, q \leq \infty$ ,  $(p, q, d) \neq (2, \infty, 2)$ , and

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

**Proposition 2.9** ([16] Proposition 3.1). *Let  $(p, q)$  and  $(a, b)$  be admissible pairs. Then, we have*

$$\begin{aligned} \|S(\cdot)\varphi\|_{L_t^p L_x^q} &\lesssim \| |\nabla|^{-2/p} \varphi \|_{L_x^2}, \\ \left\| \int_0^t S(t-t')F(t')dt' \varphi \right\|_{L_t^p L_x^q} &\lesssim \| |\nabla|^{-2/p-2/a} F \|_{L_t^{a'} L_x^{b'}}, \end{aligned}$$

where  $a'$  and  $b'$  are conjugate exponents of  $a$  and  $b$  respectively.

Propositions 2.8 and 2.9 imply the following.

**Corollary 2.10.** *Let  $(p, q)$  be an admissible pair.*

$$\|u\|_{L_t^p L_x^q} \lesssim \| |\nabla|^{-2/p} u \|_{U_S^p}, \quad u \in U_S^p. \quad (2.3)$$

Next, we define the function spaces which will be used to construct the solution. We define the projections  $P_{>1}$  and  $P_{<1}$  as

$$P_{>1} := \sum_{N \geq 1} P_N, \quad P_{<1} := Id - P_{>1}.$$

**Definition 2.11.** *Let  $s < 0$ .*

(i) *We define  $\dot{Z}^s := \{u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap U_S^2 \mid \|u\|_{\dot{Z}^s} < \infty\}$  with the norm*

$$\|u\|_{\dot{Z}^s} := \left( \sum_N N^{2s} \|P_N u\|_{U_S^2}^2 \right)^{1/2}.$$

(ii) *We define  $Z^s := \{u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \mid \|u\|_{Z^s} < \infty\}$  with the norm*

$$\|u\|_{Z^s} := \|P_{<1} u\|_{\dot{Z}^0} + \|P_{>1} u\|_{\dot{Z}^s}.$$

(iii) We define  $\dot{Y}^s := \{u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap V_S^2 \mid \|u\|_{\dot{Y}^s} < \infty\}$  with the norm

$$\|u\|_{\dot{Y}^s} := \left( \sum_N N^{2s} \|P_N u\|_{V_S^2}^2 \right)^{1/2}.$$

(iv) We define  $Y^s := \{u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \mid \|u\|_{Y^s} < \infty\}$  with the norm

$$\|u\|_{Y^s} := \|P_{<1} u\|_{\dot{Y}^0} + \|P_{>1} u\|_{\dot{Y}^s}.$$

### 3. MULTILINEAR ESTIMATE FOR $P_4(u, \bar{u}) = \bar{u}^4$ IN $1d$

In this section, we prove multilinear estimates for the nonlinearity  $\partial_x(\bar{u}^4)$  in  $1d$ , which plays a crucial role in the proof of Theorem 1.1.

**Lemma 3.1.** *We assume that  $(\tau_0, \xi_0), (\tau_1, \xi_1), \dots, (\tau_4, \xi_4) \in \mathbb{R} \times \mathbb{R}^d$  satisfy  $\sum_{j=0}^4 \tau_j = 0$  and  $\sum_{j=0}^4 \xi_j = 0$ . Then, we have*

$$\max_{0 \leq j \leq 4} |\tau_j - |\xi_j|^4| \geq \frac{1}{5} \max_{0 \leq j \leq 4} |\xi_j|^4. \quad (3.1)$$

*Proof.* By the triangle inequality, we obtain (3.1).  $\square$

#### 3.1. The homogeneous case.

**Proposition 3.2.** *Let  $d = 1$  and  $0 < T \leq \infty$ . For a dyadic number  $N_1 \in 2^{\mathbb{Z}}$ , we define the set  $A_1(N_1)$  as*

$$A_1(N_1) := \{(N_2, N_3, N_4) \in (2^{\mathbb{Z}})^3 \mid N_1 \gg N_2 \geq N_3 \geq N_4\}.$$

*If  $N_0 \sim N_1$ , then we have*

$$\begin{aligned} & \left| \sum_{A_1(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \lesssim \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^4 \|u_j\|_{\dot{Y}^{-1/2}}. \end{aligned} \quad (3.2)$$

*Proof.* We define  $u_{j,N_j,T} := \mathbf{1}_{[0,T)} P_{N_j} u_j$  ( $j = 1, \dots, 4$ ) and put  $M := N_0^4/5$ . We decompose  $Id = Q_{<M}^S + Q_{\geq M}^S$ . We divide the integrals on the left-hand side of (3.2) into 10 pieces of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 Q_j^S u_{j,N_j,T} \right) dx dt \quad (3.3)$$

with  $Q_j^S \in \{Q_{\geq M}^S, Q_{<M}^S\}$  ( $j = 0, \dots, 4$ ). By the Plancherel's theorem, we have

$$(3.3) = c \int_{\sum_{j=0}^4 \tau_j = 0} \int_{\sum_{j=0}^4 \xi_j = 0} N_0 \prod_{j=0}^4 \mathcal{F}[Q_j^S u_{j,N_j,T}](\tau_j, \xi_j),$$



where  $c$  is a constant. Therefore, Lemma 3.1 implies that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^m Q_{<M}^S u_{j,N_j,T} \right) dx dt = 0.$$

So, let us now consider the case that  $Q_j^S = Q_{\geq M}^S$  for some  $0 \leq j \leq 4$ .

First, we consider the case  $Q_0^S = Q_{\geq M}^S$ . By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \left| \sum_{A_1(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{0,N_0,T} \prod_{j=1}^4 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \leq N_0 \|Q_{\geq M}^S u_{0,N_0,T}\|_{L_{tx}^2} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \prod_{j=2}^4 \left\| \sum_{N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^6}. \end{aligned}$$

Furthermore by (2.1) and  $M \sim N_0^4$ , we have

$$\|Q_{\geq M}^S u_{0,N_0,T}\|_{L_{tx}^2} \lesssim N_0^{-2} \|u_{0,N_0,T}\|_{V_S^2}$$

and by (2.3) and  $V_S^2 \hookrightarrow U_S^4$ , we have

$$\|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \lesssim N_1^{-1/2} \|Q_1^S u_{1,N_1,T}\|_{U_S^4} \lesssim N_1^{-1/2} \|Q_1^S u_{1,N_1,T}\|_{V_S^2}.$$

While by the Sobolev inequality, (2.3),  $V_S^2 \hookrightarrow U_S^{12}$  and the Cauchy-Schwartz inequality for the dyadic sum, we have

$$\begin{aligned} \left\| \sum_{N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^6} & \lesssim \left\| |\nabla|^{1/6} \sum_{N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^3} \lesssim \left\| \sum_{N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{V_S^2} \\ & \lesssim N_1^{1/2} \left( \sum_{N_j \lesssim N_1} N_j^{-1} \|u_{j,N_j,T}\|_{V_S^2}^2 \right)^{1/2} \lesssim N_1^{1/2} \|\mathbf{1}_{[0,T)} u_j\|_{\dot{Y}^{-1/2}} \end{aligned} \quad (3.4)$$

for  $2 \leq j \leq 4$ . Therefore, we obtain

$$\begin{aligned} & \left| \sum_{A_1(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{0,N_0,T} \prod_{j=1}^m Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \lesssim \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^4 \|u_j\|_{\dot{Y}^{-1/2}} \end{aligned}$$

by (2.2) since  $\|\mathbf{1}_{[0,T)} u\|_{V_S^2} \lesssim \|u\|_{V_S^2}$  for any  $T \in (0, \infty]$ . For the case  $Q_1^S = Q_{\geq M}^S$  is proved in same way.

Next, we consider the case  $Q_i^S = Q_{\geq M}^S$  for some  $2 \leq i \leq 4$ . By the Hölder inequality, we have

$$\begin{aligned} & \left| \sum_{A_1(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{i,N_i,T} \prod_{\substack{0 \leq j \leq 4 \\ j \neq i}} Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \lesssim N_0 \|Q_0^S u_{0,N_0,T}\|_{L_t^{12} L_x^6} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \\ & \quad \times \left\| \sum_{N_i \lesssim N_1} Q_{\geq M}^S u_{i,N_i,T} \right\|_{L_{tx}^2} \prod_{\substack{2 \leq j \leq 4 \\ j \neq i}} \left\| \sum_{N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^6}. \end{aligned}$$

By  $L^2$  orthogonality and (2.1), we have

$$\begin{aligned} \left\| \sum_{N_i \lesssim N_1} Q_{\geq M}^S u_{i,N_i,T} \right\|_{L_{tx}^2} & \lesssim \left( \sum_{N_2} \|Q_{\geq M}^S u_{i,N_i,T}\|_{L_{tx}^2}^2 \right)^{1/2} \\ & \lesssim N_1^{-3/2} \|\mathbf{1}_{[0,T)} u_i\|_{\dot{Y}^{-1/2}} \end{aligned} \quad (3.5)$$

since  $M \sim N_0^4$ . While, by the calculation way as the case  $Q_0^S = Q_{\geq M}^S$ , we have

$$\begin{aligned} \|Q_0^S u_{0,N_0,T}\|_{L_t^{12} L_x^6} & \lesssim \|Q_0^S u_{0,N_0,T}\|_{V_S^2}, \\ \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} & \lesssim N_1^{-1/2} \|Q_1^S u_{1,N_1,T}\|_{V_S^2} \end{aligned}$$

and

$$\left\| \sum_{N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^6} \lesssim N_1^{1/2} \|\mathbf{1}_{[0,T)} u_j\|_{\dot{Y}^{-1/2}}.$$

Therefore, we obtain

$$\begin{aligned} & \left| \sum_{A_1(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{i,N_i,T} \prod_{\substack{0 \leq j \leq 4 \\ j \neq i}} Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \lesssim \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^4 \|u_j\|_{\dot{Y}^{-1/2}} \end{aligned}$$

by (2.2) since  $\|\mathbf{1}_{[0,T)} u\|_{V_S^2} \lesssim \|u\|_{V_S^2}$  for any  $T \in (0, \infty]$ .  $\square$

**Proposition 3.3.** *Let  $d = 1$  and  $0 < T \leq \infty$ . For a dyadic number  $N_2 \in 2^{\mathbb{Z}}$ , we define the set  $A_2(N_2)$  as*

$$A_2(N_2) := \{(N_3, N_4) \in (2^{\mathbb{Z}})^4 \mid N_2 \geq N_3 \geq N_4\}.$$

If  $N_0 \lesssim N_1 \sim N_2$ , then we have

$$\begin{aligned} & \left| \sum_{A_2(N_2)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \lesssim \frac{N_0}{N_1} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} N_2^{-1/2} \|P_{N_2} u_2\|_{V_S^2} \|u_3\|_{\dot{Y}^{-1/2}} \|u_4\|_{\dot{Y}^{-1/2}}. \end{aligned} \quad (3.6)$$

The proof of Proposition 3.3 is quite similar as the proof of Proposition 3.2.

### 3.2. The inhomogeneous case.

**Proposition 3.4.** *Let  $d = 1$  and  $0 < T \leq 1$ . For a dyadic number  $N_1 \in 2^{\mathbb{Z}}$ , we define the set  $A'_1(N_1)$  as*

$$A'_1(N_1) := \{(N_2, N_3, N_4) \in (2^{\mathbb{Z}})^3 \mid N_1 \gg N_2 \geq N_3 \geq N_4, N_4 \leq 1\}.$$

If  $N_0 \sim N_1$ , then we have

$$\left| \sum_{A'_1(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \lesssim T^{\frac{1}{6}} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^4 \|u_j\|_{Y^{-1/2}}. \quad (3.7)$$

*Proof.* We further divide  $A'_1(N_1)$  into three pieces:

$$\begin{aligned} A'_1(N_1) &= \bigcup_{j=1}^3 A'_{1,j}(N_1), \\ A'_{1,1}(N_1) &:= \{(N_2, N_3, N_4) \in A'_1(N_1) : N_3 \geq 1\}, \\ A'_{1,2}(N_1) &:= \{(N_2, N_3, N_4) \in A'_1(N_1) : N_2 \geq 1 \geq N_3\}, \\ A'_{1,3}(N_1) &:= \{(N_2, N_3, N_4) \in A'_1(N_1) : 1 \geq N_2\}. \end{aligned}$$

We define  $u_{j,N_j} := P_{N_j} u_j$ ,  $u_{j,T} := \mathbf{1}_{[0,T)} u_j$  and  $u_{j,N_j,T} := \mathbf{1}_{[0,T)} P_{N_j} u_j$  ( $j = 1, \dots, 4$ ). We firstly consider the case  $A'_{1,1}(N_1)$ . In the case  $T \leq N_0^{-3}$ , the Hölder inequality implies

$$\begin{aligned} & \left| \sum_{A'_{1,1}(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \leq N_0 \|\mathbf{1}_{[0,T)}\|_{L_t^2} \|u_{0,N_0}\|_{L_t^4 L_x^\infty} \|u_{1,N_1}\|_{L_t^4 L_x^\infty} \prod_{j=2}^3 \left\| \sum_{1 \leq N_j \leq N_1} u_{j,N_j} \right\|_{L_t^\infty L_x^2} \|P_{<1} u_4\|_{L_t^\infty L_x^\infty} \end{aligned}$$

Furthermore by (2.3) and  $V_S^2 \hookrightarrow U_S^4$ , we have

$$\begin{aligned} \|u_{0,N_0}\|_{L_t^4 L_x^\infty} \|u_{1,N_1}\|_{L_t^4 L_x^\infty} &\lesssim N_0^{-1/2} \|u_{0,N_0}\|_{U_S^4} N_1^{-1/2} \|Q_1^S u_{1,N_1}\|_{U_S^4} \\ &\lesssim N_0^{-1} \|u_{0,N_0}\|_{V_S^2} \|u_{1,N_1}\|_{V_S^2} \end{aligned}$$

and by the Sobolev inequality,  $V_S^2 \hookrightarrow L_t^\infty L_x^2$  and the Cauchy-Schwartz inequality, we have

$$\|P_{<1}u_4\|_{L_t^\infty L_x^\infty} \lesssim \|P_{<1}u_4\|_{L_t^\infty L_x^2} \lesssim \left( \sum_{N \leq 2} \|P_N P_{<1}u_4\|_{V_S^2}^2 \right)^{1/2} \leq \|P_{<1}u_4\|_{\dot{Y}^0}$$

While by  $L^2$  orthogonality and  $V_S^2 \hookrightarrow L_t^\infty L_x^2$ , we have

$$\left\| \sum_{1 \leq N_j \leq N_1} u_{j,N_j} \right\|_{L_t^\infty L_x^2} \lesssim \left( \sum_{1 \leq N_j \leq N_1} \|u_{j,N_j}\|_{V_S^2}^2 \right)^{1/2} \lesssim N_0^{1/2} \|P_{>1}u_j\|_{\dot{Y}^{-1/2}}$$

Therefore, we obtain

$$\begin{aligned} & \left| \sum_{A'_{1,1}(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \lesssim T^{1/2} N_0 \|u_{0,N_0}\|_{V_S^2} \|u_{1,N_1}\|_{V_S^2} \prod_{j=2}^3 \|P_{>1}u_j\|_{\dot{Y}^{-1/2}} \|P_{<1}u_4\|_{\dot{Y}^0} \end{aligned}$$

and note that  $T^{1/2}N_0 \leq T^{1/6}$ .

In the case  $T \geq N_0^{-3}$ , we divide the integrals on the left-hand side of (3.2) into 10 pieces of the form (3.3) in the proof of Proposition 3.2. Thanks to Lemma 3.1, let us consider the case that  $Q_j^S = Q_{\geq M}^S$  for some  $0 \leq j \leq 4$ . First, we consider the case  $Q_0^S = Q_{\geq M}^S$ . By the same way as in the proof of Proposition 3.2 and using

$$\|Q_4^S P_{<1}u_{4,T}\|_{L_t^{12} L_x^6} \lesssim \|Q_4^S P_{<1}u_{4,T}\|_{V_S^2} \lesssim \|P_{<1}u_{4,T}\|_{\dot{Y}^0}$$

instead of (3.4), we obtain

$$\begin{aligned} & \left| \sum_{A'_{1,1}(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{0,N_0,T} \prod_{j=1}^4 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \leq N_0 \|Q_{\geq M}^S u_{0,N_0,T}\|_{L_{tx}^2} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \prod_{j=2}^3 \left\| \sum_{1 \leq N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^6} \|Q_4^S P_{<1}u_{4,T}\|_{L_t^{12} L_x^6} \\ & \lesssim N_0^{-\frac{1}{2}} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^3 \|P_{>1}u_j\|_{\dot{Y}^{-1/2}} \|P_{<1}u_4\|_{\dot{Y}^0} \end{aligned}$$

and note that  $N_0^{-1/2} \leq T^{1/6}$ . Since the cases  $Q_j^S = Q_{\geq M}^S$  ( $j = 1, 2, 3$ ) are similarly handled, we omit the details here.

We focus on the case  $Q_4^S = Q_{\geq M}^S$ . By the same way as in the proof of Proposition 3.2 and using

$$\|Q_{\geq M}^S P_{<1} u_{4,T}\|_{L_{tx}^2} \lesssim N_0^{-2} \|P_{<1} u_{4,T}\|_{V_S^2} \lesssim N_0^{-2} \|P_{<1} u_{4,T}\|_{\dot{Y}^0}$$

instead of (3.5) with  $j = 4$ , we obtain

$$\begin{aligned} & \left| \sum_{A'_{1,1}(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{4,N_4,T} \prod_{j=0}^3 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \leq N_0 \|u_{0,N_0,T}\|_{L_t^{12} L_x^6} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \prod_{j=2}^3 \left\| \sum_{1 \leq N_j \lesssim N_1} Q_j^S u_{j,N_j,T} \right\|_{L_t^{12} L_x^6} \|Q_{\geq M}^S P_{<1} u_{4,T}\|_{L_{tx}^2} \\ & \lesssim N_0^{-1/2} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^3 \|P_{>1} u_j\|_{\dot{Y}^{-1/2}} \|P_{<1} u_4\|_{\dot{Y}^0} \end{aligned}$$

and note that  $N_0^{-1/2} \leq T^{1/6}$ .

We secondly consider the case  $A'_{1,2}(N_1)$ . In the case  $T \leq N_0^{-3}$ , the Hölder inequality implies

$$\begin{aligned} & \left| \sum_{A'_{1,2}(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \leq N_0 \|\mathbf{1}_{[0,T]}\|_{L_t^2} \|u_{0,N_0}\|_{L_t^4 L_x^\infty} \|u_{1,N_1}\|_{L_t^4 L_x^\infty} \left\| \sum_{1 \leq N_2 \lesssim N_1} u_{2,N_2} \right\|_{L_t^\infty L_x^2} \prod_{j=3}^4 \|P_{<1} u_j\|_{L_t^\infty L_x^4}. \end{aligned}$$

By the same estimates as in the proof for the case  $A'_{1,1}(N_1)$  and

$$\|P_{<1} u_j\|_{L_t^\infty L_x^4} \lesssim \|P_{<1} u_j\|_{L_t^\infty L_x^2} \lesssim \left( \sum_{N \leq 2} \|P_N P_{<1} u_j\|_{V_S^2}^2 \right)^{1/2} \leq \|P_{<1} u_j\|_{\dot{Y}^0}$$

for  $j = 3, 4$ , we obtain

$$\begin{aligned} & \left| \sum_{A'_{1,2}(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \lesssim T^{1/2} N_0^{1/2} \|u_{0,N_0}\|_{V_S^2} \|u_{1,N_1}\|_{V_S^2} \|P_{>1} u_2\|_{\dot{Y}^{-1/2}} \prod_{j=3}^4 \|P_{<1} u_j\|_{\dot{Y}^0} \end{aligned}$$

and note that  $T^{1/2} N_0^{1/2} \leq T^{1/3}$ .

In the case  $T \geq N_0^{-3}$ , we divide the integrals on the left-hand side of (3.2) into 10 pieces of the form (3.3) in the proof of Proposition 3.2. Thanks to Lemma 3.1, let

us consider the case that  $Q_j^S = Q_{\geq M}^S$  for some  $0 \leq j \leq 4$ . By the same argument as in the proof for the case  $A'_{1,1}(N_1)$ , we obtain

$$\begin{aligned}
& \left| \sum_{A'_{1,2}(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{0,N_0,T} \prod_{j=1}^4 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\
& \leq N_0 \|Q_{\geq M}^S u_{0,N_0,T}\|_{L_{tx}^2} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \left\| \sum_{1 \leq N_2 \lesssim N_1} Q_2^S u_{2,N_2,T} \right\|_{L_t^{12} L_x^6} \prod_{j=3}^4 \|Q_j^S P_{<1} u_{j,T}\|_{L_t^{12} L_x^6} \\
& \lesssim N_0^{-1} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \|P_{>1} u_2\|_{\dot{Y}^{-1/2}} \prod_{j=3}^4 \|P_{<1} v_j\|_{\dot{Y}^0}
\end{aligned}$$

if  $Q_0 = Q_{\geq M}^S$  and

$$\begin{aligned}
& \left| \sum_{A'_{1,2}(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_4 Q_{\geq M}^S u_{4,N_4,T} \prod_{j=0}^3 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\
& \leq N_0 \|u_{0,N_0,T}\|_{L_t^{12} L_x^6} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \left\| \sum_{1 \leq N_2 \lesssim N_1} Q_2^S u_{2,N_2,T} \right\|_{L_t^{12} L_x^6} \\
& \quad \times \|Q_3^S P_{<1} u_{3,T}\|_{L_t^{12} L_x^6} \|Q_{\geq M}^S P_{<1} u_{4,T}\|_{L_{tx}^2} \\
& \lesssim N_0^{-1} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \|P_{>1} u_2\|_{\dot{Y}^{\frac{1}{2}}} \prod_{j=3}^4 \|P_{<1} u_j\|_{\dot{Y}^0}
\end{aligned}$$

if  $Q_4 = Q_{\geq M}^S$ . Note that  $N_0^{-1} \leq T^{1/3}$ . The remaining cases follow from the same argument as above.

We thirdly consider the case  $A'_{1,3}(N_1)$ . In the case  $T \leq N_0^{-3}$ , the Hölder inequality implies

$$\begin{aligned}
& \left| \sum_{A'_{1,3}(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\
& \leq N_0 \|\mathbf{1}_{[0,T]}\|_{L_t^2} \|u_{0,N_0}\|_{L_t^4 L_x^\infty} \|u_{1,N_1}\|_{L_t^4 L_x^\infty} \prod_{j=2}^4 \|P_{<1} u_j\|_{L_t^\infty L_x^3}.
\end{aligned}$$

By the same estimates as in the proof for the case  $A'_{1,1}(N_1)$  and

$$\|P_{<1} u_j\|_{L_t^\infty L_x^3} \lesssim \|P_{<1} u_j\|_{L_t^\infty L_x^2} \lesssim \left( \sum_{N \leq 2} \|P_N P_{<1} u_j\|_{V_S^2}^2 \right)^{1/2} \leq \|P_{<1} u_j\|_{\dot{Y}^0}$$

for  $j = 2, 3, 4$ , we obtain

$$\left| \sum_{A'_{1,3}(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \lesssim T^{1/2} \|u_{0,N_0}\|_{V_S^2} \|u_{1,N_1}\|_{V_S^2} \prod_{j=2}^4 \|P_{<1} u_j\|_{\dot{Y}^0}.$$

In the case  $T \geq N_0^{-3}$ , we divide the integrals on the left-hand side of (3.2) into 10 pieces of the form (3.3) in the proof of Proposition 3.2. Thanks to Lemma 3.1, let us consider the case that  $Q_j^S = Q_{\geq M}^S$  for some  $0 \leq j \leq 4$ . By the same argument as in the proof for the case  $A'_{1,1}(N_1)$ , we obtain

$$\begin{aligned} & \left| \sum_{A'_{1,3}(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_0 Q_{\geq M}^S u_{0,N_0,T} \prod_{j=1}^4 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \leq N_0 \|Q_{\geq M}^S u_{0,N_0,T}\|_{L_{tx}^2} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \prod_{j=2}^4 \|Q_j^S P_{<1} u_{j,T}\|_{L_t^{12} L_x^6} \\ & \lesssim N_0^{-3/2} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \|P_{<1} u_2\|_{Y^{-1/2}} \prod_{j=3}^4 \|P_{<1} v_j\|_{\dot{Y}^0} \end{aligned}$$

if  $Q_0 = Q_{\geq M}^S$  and

$$\begin{aligned} & \left| \sum_{A'_{1,3}(N_1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( N_4 Q_{\geq M}^S u_{4,N_4,T} \prod_{j=0}^3 Q_j^S u_{j,N_j,T} \right) dx dt \right| \\ & \leq N_0 \|u_{0,N_0,T}\|_{L_t^{12} L_x^6} \|Q_1^S u_{1,N_1,T}\|_{L_t^4 L_x^\infty} \prod_{j=2}^3 \|Q_j^S P_{<1} u_{j,T}\|_{L_t^{12} L_x^6} \|Q_{\geq M}^S P_{<1} u_{4,T}\|_{L_{tx}^2} \\ & \lesssim N_0^{-3/2} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^4 \|P_{<1} u_j\|_{Y^0} \end{aligned}$$

if  $Q_4 = Q_{\geq M}^S$ . Note that  $N_0^{-3/2} \leq T^{1/2}$ . The cases  $Q_j^S = Q_{\geq M}^S$  ( $j = 1, 2, 3$ ) are the same argument as above. □

Furthermore, we obtain the following estimate.

**Proposition 3.5.** *Let  $d = 1$  and  $0 < T \leq 1$ . For a dyadic number  $N_2 \in 2^{\mathbb{Z}}$ , we define the set  $A'_2(N_2)$  as*

$$A'_2(N_2) := \{(N_3, N_4) \in (2^{\mathbb{Z}})^4 \mid N_2 \geq N_3 \geq N_4, N_4 \leq 1\}.$$

If  $N_0 \lesssim N_1 \sim N_2$ , then we have

$$\begin{aligned} & \left| \sum_{A'_2(N_2)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right| \\ & \lesssim T^{\frac{1}{6}} \frac{N_0}{N_1} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} N_2^{-1/2} \|P_{N_2} u_2\|_{V_S^2} \|u_3\|_{Y^{-1/2}} \|u_4\|_{Y^{-1/2}}. \end{aligned} \quad (3.8)$$

Because the proof is similar as above, we skip the proof.

#### 4. PROOF OF WELL-POSEDNESS

**4.1. The small data case.** In this section, we prove Theorem 1.1 and Corollary 1.3. We define the map  $\Phi_{T,\varphi}$  as

$$\Phi_{T,\varphi}(u)(t) := S(t)\varphi - iI_T(u, \dots, u)(t),$$

where

$$I_T(u_1, \dots, u_4)(t) := \int_0^t \mathbf{1}_{[0,T)}(t') S(t-t') \partial_x \left( \prod_{j=1}^4 \overline{u_j(t')} \right) dt'.$$

To prove the well-posedness of (1.1) in  $\dot{H}^{-1/2}$ , we prove that  $\Phi_{T,\varphi}$  is a contraction map on a closed subset of  $\dot{Z}^{-1/2}([0, T))$ . Key estimate is the following:

**Proposition 4.1.** *Let  $d = 1$ . For any  $0 < T < \infty$ , we have*

$$\|I_T(u_1, \dots, u_4)\|_{\dot{Z}^{-1/2}} \lesssim \prod_{j=1}^4 \|u_j\|_{\dot{Y}^{-1/2}}. \quad (4.1)$$

*Proof.* We decompose

$$I_T(u_1, \dots, u_m) = \sum_{N_1, \dots, N_4} I_T(P_{N_1} u_1, \dots, P_{N_4} u_4).$$

By symmetry, it is enough to consider the summation for  $N_1 \geq N_2 \geq N_3 \geq N_4$ . We put

$$\begin{aligned} S_1 &:= \{(N_1, \dots, N_m) \in (2^{\mathbb{Z}})^m \mid N_1 \gg N_2 \geq N_3 \geq N_4\} \\ S_2 &:= \{(N_1, \dots, N_m) \in (2^{\mathbb{Z}})^m \mid N_1 \sim N_2 \geq N_3 \geq N_4\} \end{aligned}$$

and

$$J_k := \left\| \sum_{S_k} I_T(P_{N_1} u_1, \dots, P_{N_4} u_4) \right\|_{\dot{Z}^{-1/2}} \quad (k = 1, 2).$$



First, we prove the estimate for  $J_1$ . By Theorem 2.4 and the Plancherel's theorem, we have

$$\begin{aligned} J_1 &\leq \left\{ \sum_{N_0} N_0^{-1} \left\| S(-\cdot) P_{N_0} \sum_{S_1} I_T(P_{N_1} u_1, \dots, P_{N_4} u_4) \right\|_{U^2}^2 \right\}^{1/2} \\ &\lesssim \left\{ \sum_{N_0} N_0^{-1} \sum_{N_1 \sim N_0} \left( \sup_{\|u_0\|_{V_S^2}=1} \left| \sum_{A_1(N_1)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right|^2 \right)^{1/2} \right\}, \end{aligned}$$

where  $A_1(N_1)$  is defined in Proposition 3.2. Therefore by Proposition 3.2, we have

$$\begin{aligned} J_1 &\lesssim \left\{ \sum_{N_0} N_0^{-1} \sum_{N_1 \sim N_0} \left( \sup_{\|u_0\|_{V_S^2}=1} \|P_{N_0} u_0\|_{V_S^2} \|P_{N_1} u_1\|_{V_S^2} \prod_{j=2}^4 \|u_j\|_{\dot{Y}^{-1/2}} \right)^2 \right\}^{1/2} \\ &\lesssim \left( \sum_{N_1} N_1^{-1} \|P_{N_1} u_1\|_{V_{\Delta}^2}^2 \right)^{1/2} \prod_{j=2}^4 \|u_j\|_{\dot{Y}^{-1/2}} \\ &= \prod_{j=1}^4 \|u_j\|_{\dot{Y}^{-1/2}}. \end{aligned}$$

Next, we prove the estimate for  $J_2$ . By Theorem 2.4 and the Plancherel's theorem, we have

$$\begin{aligned} J_2 &\leq \sum_{N_1} \sum_{N_2 \sim N_1} \left( \sum_{N_0} N_0^{-1} \left\| S(-\cdot) P_{N_0} \sum_{A_2(N_2)} I_T(P_{N_1} u_1, \dots, P_{N_4} u_4) \right\|_{U^2}^2 \right)^{1/2} \\ &= \sum_{N_1} \sum_{N_2 \sim N_1} \left( \sum_{N_0 \lesssim N_1} N_0^{-1} \sup_{\|u_0\|_{V_S^2}=1} \left| \sum_{A_2(N_2)} \int_0^T \int_{\mathbb{R}} \left( N_0 \prod_{j=0}^4 P_{N_j} u_j \right) dx dt \right|^2 \right)^{1/2}, \end{aligned}$$

where  $A_2(N_2)$  is defined in Proposition 3.3. Therefore by Proposition 3.3 and Cauchy-Schwartz inequality for the dyadic sum, we have

$$\begin{aligned} J_2 &\lesssim \sum_{N_1} \sum_{N_2 \sim N_1} \left( \sum_{N_0 \lesssim N_1} N_0^{-1} \left( \frac{N_0}{N_1} \|P_{N_1} u_1\|_{V_S^2} N_2^{-1/2} \|P_{N_2} u_2\|_{V_S^2} \|u_3\|_{\dot{Y}^{-1/2}} \|u_4\|_{\dot{Y}^{-1/2}} \right)^2 \right)^{1/2} \\ &\lesssim \left( \sum_{N_1} N_1^{-1} \|P_{N_1} u_1\|_{V_S^2}^2 \right)^{1/2} \left( \sum_{N_2} N_2^{-1} \|P_{N_2} u_2\|_{V_S^2}^2 \right)^{1/2} \|u_3\|_{\dot{Y}^{-1/2}} \|u_4\|_{\dot{Y}^{-1/2}} \\ &= \prod_{j=1}^4 \|u_j\|_{\dot{Y}^{sc}}. \end{aligned}$$

□

**Proof of Theorem 1.1.** For  $r > 0$ , we define

$$\dot{Z}_r^s(I) := \left\{ u \in \dot{Z}^s(I) \mid \|u\|_{\dot{Z}^s(I)} \leq 2r \right\} \quad (4.2)$$

which is a closed subset of  $\dot{Z}^s(I)$ . Let  $T > 0$  and  $u_0 \in B_r(\dot{H}^{-1/2})$  are given. For  $u \in \dot{Z}_r^{-1/2}([0, T])$ , we have

$$\|\Phi_{T,u_0}(u)\|_{\dot{Z}^{-1/2}([0,T])} \leq \|u_0\|_{\dot{H}^{-1/2}} + C\|u\|_{\dot{Z}^{-1/2}([0,T])}^4 \leq r(1 + 16Cr^3)$$

and

$$\begin{aligned} \|\Phi_{T,u_0}(u) - \Phi_{T,u_0}(v)\|_{\dot{Z}^{-1/2}([0,T])} &\leq C(\|u\|_{\dot{Z}^{-1/2}([0,T])} + \|v\|_{\dot{Z}^{-1/2}([0,T])})^3 \|u - v\|_{\dot{Z}^{-1/2}([0,T])} \\ &\leq 64Cr^3 \|u - v\|_{\dot{Z}^{-1/2}([0,T])} \end{aligned}$$

by Proposition 4.1 and

$$\|S(\cdot)u_0\|_{\dot{Z}^{-1/2}([0,T])} \leq \|\mathbf{1}_{[0,T]}S(\cdot)u_0\|_{\dot{Z}^{-1/2}} \leq \|u_0\|_{\dot{H}^{-1/2}},$$

where  $C$  is an implicit constant in (4.1). Therefore if we choose  $r$  satisfying

$$r < (64C)^{-1/3},$$

then  $\Phi_{T,u_0}$  is a contraction map on  $\dot{Z}_r^{-1/2}([0, T])$ . This implies the existence of the solution of (1.1) and the uniqueness in the ball  $\dot{Z}_r^{-1/2}([0, T])$ . The Lipschitz continuously of the flow map is also proved by similar argument.  $\square$

Corollary 1.3 is obtained by the same way as the proof of Corollary 1.2 in [10].

**4.2. The large data case.** In this subsection, we prove Theorem 1.4. The following is the key estimate.

**Proposition 4.2.** *Let  $d = 1$ . We have*

$$\|I_1(u_1, \dots, u_4)\|_{\dot{Z}^{-1/2}} \lesssim \prod_{j=1}^4 \|u_j\|_{Y^{-1/2}}. \quad (4.3)$$

*Proof.* We decompose  $u_j = v_j + w_j$  with  $v_j = P_{>1}u_j \in \dot{Y}^{-1/2}$  and  $w_j = P_{<1}u_j \in \dot{Y}^0$ . From Propositions 3.4, 3.5, and the same way as in the proof of Proposition 4.1, it remains to prove that

$$\|I_1(w_1, w_2, w_3, w_4)\|_{\dot{Z}^{-1/2}} \lesssim \prod_{j=1}^4 \|w_j\|_{\dot{Y}^0}.$$

By Theorem 2.4, the Cauchy-Schwartz inequality, the Hölder inequality and the Sobolev inequality, we have

$$\|I_1(w_1, w_2, w_3, w_4)\|_{\dot{Z}^{-1/2}} \lesssim \left\| \prod_{j=1}^4 \overline{w_j} \right\|_{L^1([0,1];L^2)} \lesssim \prod_{j=1}^4 \|w_j\|_{L_t^\infty L_x^2} \lesssim \prod_{j=1}^4 \|w_j\|_{\dot{Y}^0},$$

which completes the proof.  $\square$

**Proof of Theorem 1.4.** Let  $u_0 \in B_{\delta,R}(H^{-1/2})$  with  $u_0 = v_0 + w_0$ ,  $v_0 \in \dot{H}^{-1/2}$ ,  $w_0 \in L^2$ . A direct calculation yields

$$\|S(t)u_0\|_{Z^{-1/2}([0,1])} \leq \delta + R.$$

We start with the case  $R = \delta = (4C + 4)^{-4}$ , where  $C$  is the implicit constant in (4.3). Proposition 4.2 implies that for  $u \in Z_r^{-1/2}([0,1])$  with  $r = 1/(4C + 4)$

$$\begin{aligned} \|\Phi_{1,u_0}(u)\|_{Z^{-1/2}([0,1])} &\leq \|S(t)u_0\|_{Z^{-1/2}([0,1])} + C\|u\|_{Z^{-1/2}([0,1])}^4 \\ &\leq 2r^4 + 16Cr^4 = r^4(16C + 2) \leq r \end{aligned}$$

and

$$\begin{aligned} \|\Phi_{1,u_0}(u) - \Phi_{1,u_0}(v)\|_{Z^{-1/2}([0,1])} &\leq C(\|u\|_{Z^{-1/2}([0,1])} + \|v\|_{Z^{-1/2}([0,1])})^3 \|u - v\|_{Z^{-1/2}([0,1])} \\ &\leq 64Cr^3 \|u - v\|_{Z^{-1/2}([0,1])} < \|u - v\|_{Z^{-1/2}([0,1])} \end{aligned}$$

if we choose  $C$  large enough (namely,  $r$  is small enough). Accordingly,  $\Phi_{1,u_0}$  is a contraction map on  $\dot{Z}_r^{-1/2}([0,1])$ .

We note that all of the above remains valid if we exchange  $Z^{-1/2}([0,1])$  by the smaller space  $\dot{Z}^{-1/2}([0,1])$  since  $\dot{Z}^{-1/2}([0,1]) \hookrightarrow Z^{-1/2}([0,1])$  and the left hand side of (4.3) is the homogeneous norm.

We now assume that  $u_0 \in B_{\delta,R}(H^{-1/2})$  for  $R \geq \delta = (4C + 4)^{-4}$ . We define  $u_{0,\lambda}(x) = \lambda^{-1}u_0(\lambda^{-1}x)$ . For  $\lambda = \delta^{-2}R^2$ , we observe that  $u_{0,\lambda} \in B_{\delta,\delta}(H^{-1/2})$ . We therefore find a solution  $u_\lambda \in Z^{-1/2}([0,1])$  with  $u_\lambda(0, x) = u_{0,\lambda}(x)$ . By the scaling, we find a solution  $u \in Z^{-1/2}([0, \delta^8 R^{-8}])$ .

Thanks to Propositions 3.4 and 3.5, the uniqueness follows from the same argument as in [5].  $\square$

## 5. PROOF OF THEOREM 1.6

In this section, we prove Theorem 1.6. We only prove for the homogeneous case since the proof for the inhomogeneous case is similar. We define the map  $\Phi_{T,\varphi}^m$  as

$$\Phi_{T,\varphi}^m(u)(t) := S(t)\varphi - iI_T^m(u, \dots, u)(t),$$

where

$$I_T^m(u_1, \dots, u_m)(t) := \int_0^t \mathbf{1}_{[0,T)}(t') S(t - t') \partial \left( \prod_{j=1}^m u_j(t') \right) dt'.$$

and the solution space  $\dot{X}^s$  as

$$\dot{X}^s := C(\mathbb{R}; \dot{H}^s) \cap L^{p_m}(\mathbb{R}; \dot{W}^{s+1/(m-1), q_m}),$$

where  $p_m = 2(m-1)$ ,  $q_m = 2(m-1)d/\{(m-1)d-2\}$  for  $d \geq 2$  and  $p_3 = 4$ ,  $q_3 = \infty$  for  $d = 1$ . To prove the well-posedness of (1.1) in  $L^2(\mathbb{R})$  or  $H^{s_c}(\mathbb{R}^d)$ , we prove that  $\Phi_{T,\varphi}$  is a contraction map on a closed subset of  $\dot{X}^s$ . The key estimate is the following:

**Proposition 5.1.** (i) *Let  $d = 1$  and  $m = 3$ . For any  $0 < T < \infty$ , we have*

$$\|I_T^3(u_1, u_2, u_3)\|_{\dot{X}^0} \lesssim T^{1/2} \prod_{j=1}^3 \|u_j\|_{\dot{X}^0}. \quad (5.1)$$

(ii) *Let  $d \geq 2$ ,  $(m-1)d \geq 4$  and  $s_c = d/2 - 3/(m-1)$ . For any  $0 < T \leq \infty$ , we have*

$$\|I_T^m(u_1, \dots, u_m)\|_{\dot{X}^{s_c}} \lesssim \prod_{j=1}^m \|u_j\|_{\dot{X}^{s_c}}. \quad (5.2)$$

*Proof.* (i) By Proposition 2.9 with  $(a, b) = (4, \infty)$ , we get

$$\|I_T^3(u_1, u_2, u_3)\|_{L_t^\infty L_x^2} \lesssim \left\| \mathbf{1}_{[0,T]} |\nabla|^{-1/2} \partial \left( \prod_{j=1}^3 u_j \right) \right\|_{L_t^{4/3} L_x^1}$$

and

$$\| |\nabla|^{1/2} I_T^3(u_1, u_2, u_3) \|_{L_t^4 L_x^\infty} \lesssim \left\| \mathbf{1}_{[0,T]} |\nabla|^{1/2-1/2-1/2} \partial \left( \prod_{j=1}^3 u_j \right) \right\|_{L_t^{4/3} L_x^1}.$$

Therefore, thanks to the fractional Leibniz rule (see [1]), we have

$$\begin{aligned} \|I_T^3(u_1, \dots, u_3)\|_{\dot{X}^0} &\lesssim \left\| \mathbf{1}_{[0,T]} |\nabla|^{1/2} \prod_{j=1}^3 u_j \right\|_{L_t^{4/3} L_x^1} \\ &\lesssim \|\mathbf{1}_{[0,T]}\|_{L_t^2} \| |\nabla|^{1/2} u_i \|_{L_t^4 L_x^\infty} \prod_{\substack{1 \leq j \leq 3 \\ j \neq i}} \|u_j\|_{L_t^\infty L_x^2} \\ &\lesssim T^{1/2} \prod_{j=1}^3 \|u_j\|_{\dot{X}^0} \end{aligned}$$

by the Hölder inequality.

(ii) By Proposition 2.9 with

$$(a, b) = \left( \frac{2(m-1)}{m-2}, \frac{2(m-1)d}{(m-1)d-2(m-2)} \right), \quad (5.3)$$

we get

$$\| |\nabla|^{s_c} I_T^m(u_1, \dots, u_m) \|_{L_t^\infty L_x^2} \lesssim \left\| |\nabla|^{s_c-2/a} \partial \left( \prod_{j=1}^m u_j \right) \right\|_{L_t^{a'} L_x^{b'}}$$

and

$$\| |\nabla|^{s_c+1/(m-1)} I_T^m(u_1, \dots, u_m) \|_{L_t^{p_m} L_x^{q_m}} \lesssim \left\| |\nabla|^{s_c+1/(m-1)-2/p_m-2/a} \partial \left( \prod_{j=1}^m u_j \right) \right\|_{L_t^{a'} L_x^{b'}}.$$

Therefore, thanks to the fractional Leibniz rule (see [1]), we have

$$\begin{aligned} \| I_T^m(u_1, \dots, u_m) \|_{\dot{X}^{s_c}} &\lesssim \left\| |\nabla|^{s_c+1/(m-1)} \prod_{j=1}^m u_j \right\|_{L_t^{a'} L_x^{b'}} \\ &\lesssim \sum_{i=1}^m \| |\nabla|^{s_c+1/(m-1)} u_i \|_{L_t^{p_m} L_x^{q_m}} \prod_{\substack{1 \leq j \leq m \\ j \neq i}} \| u_j \|_{L_t^{p_m} L_x^{(m-1)d}} \\ &\lesssim \sum_{i=1}^m \| |\nabla|^{s_c+1/(m-1)} u_i \|_{L_t^{p_m} L_x^{q_m}} \prod_{\substack{1 \leq j \leq m \\ j \neq i}} \| |\nabla|^{s_c+1/(m-1)} u_j \|_{L_t^{p_m} L_x^{q_m}} \\ &\lesssim \prod_{j=1}^m \| u_j \|_{\dot{X}^{s_c}} \end{aligned}$$

by the Hölder inequality and the Sobolev inequality, where we used the condition  $(m-1)d \geq 4$  which is equivalent to  $s_c + 1/(m-1) \geq 0$ .  $\square$

The well-posedness can be proved by the same way as the proof of Theorem 1.1 and the scattering follows from that the Strichartz estimate because the  $\dot{X}^{s_c}$  norm of the nonlinear part is bounded by the norm of the  $L^{p_m} L^{q_m}$  space (see for example [16, Section 9]).

## 6. PROOF OF THEOREM 1.8

In this section we prove the flow of (1.1) is not smooth. Let  $u^{(m)}[u_0]$  be the  $m$ -th iteration of (1.1) with initial data  $u_0$ :

$$u^{(m)}[u_0](t, x) := -i \int_0^t e^{i(t-t')\Delta^2} \partial P_m(S(t')u_0, S(-t')\overline{u_0}) dt'.$$

Firstly we consider the case  $d = 1$ ,  $m = 3$ ,  $P_3(u, \overline{u}) = |u|^2 u$ . For  $N \gg 1$ , we put

$$f_N = N^{-s+1/2} \mathcal{F}^{-1}[\mathbf{1}_{[N-N^{-1}, N+N^{-1}]}]$$

Let  $u_N^{(3)}$  be the third iteration of (1.1) for the data  $f_N$ . Namely,

$$u_N^{(3)}(t, x) = u^{(3)}[f_N](t, x) = -i \int_0^t e^{i(t-t')\partial_x^4} \partial_x \left( |e^{it'\partial_x^4} f_N|^2 e^{it'\partial_x^4} f_N \right) (x) dt'.$$

Note that  $\|f_N\|_{H^s} \sim 1$ . Theorem 1.8 is implied by the following propositions.

**Proposition 6.1.** *If  $s < 0$ , then for any  $N \gg 1$ , we have*

$$\|u_N^{(3)}\|_{L^\infty([0,1];H^s)} \rightarrow \infty$$

as  $N \rightarrow \infty$ .

*Proof.* A direct calculation implies

$$\widehat{u_N^{(3)}}(t, \xi) = e^{it\xi^4} \xi \int_{\xi_1 - \xi_2 + \xi_3 = \xi} \int_0^t e^{it'(-\xi^4 + \xi_1^4 - \xi_2^4 + \xi_3^4)} dt' \widehat{f_N}(\xi_1) \overline{\widehat{f_N}(\xi_2)} \widehat{f_N}(\xi_3)$$

and

$$\begin{aligned} & -(\xi_1 - \xi_2 + \xi_3)^4 + \xi_1^4 - \xi_2^4 + \xi_3^4 \\ & = 2(\xi_1 - \xi_2)(\xi_2 - \xi_3)(2\xi_1^2 + \xi_2^2 + 2\xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 + 3\xi_3\xi_1). \end{aligned} \tag{6.1}$$

From  $\xi_j \in [N - N^{-1}, N + N^{-1}]$  for  $j = 1, 2, 3$ , we get

$$|-(\xi_1 - \xi_2 + \xi_3)^4 + \xi_1^4 - \xi_2^4 + \xi_3^4| \lesssim 1.$$

We therefore obtain for sufficiently small  $t > 0$

$$\begin{aligned} |\widehat{u_N^{(3)}}(t, \xi)| & \gtrsim tN^{-3s+5/2} \left| \int_{\xi_1 - \xi_2 + \xi_3 = \xi} \mathbf{1}_{[N-N^{-1}, N+N^{-1}]}(\xi_1) \mathbf{1}_{[N-N^{-1}, N+N^{-1}]}(\xi_2) \mathbf{1}_{[N-N^{-1}, N+N^{-1}]}(\xi_3) \right| \\ & \gtrsim tN^{-3s+1/2} \mathbf{1}_{[N-N^{-1}, N+N^{-1}]}(\xi). \end{aligned}$$

Hence,

$$\|u_N^{(3)}\|_{L^\infty([0,1];H^s)} \gtrsim N^{-2s}.$$

This lower bound goes to infinity as  $N$  tends to infinity if  $s < 0$ , which concludes the proof.  $\square$

Secondly, we show that absence of a smooth flow map for  $d \geq 1$  and  $m \geq 2$ . Putting

$$g_N := N^{-s-d/2} \mathcal{F}^{-1}[\mathbf{1}_{[-N,N]^d}],$$

we set  $u_N^{(m)} := u^{(m)}[g_N]$ . Note that  $\|g_N\|_{H^s} \sim 1$ . As above, we show the following.

**Proposition 6.2.** *If  $s < s_c := d/2 - 3/(m-1)$  and  $\partial = |\nabla|$  or  $\frac{\partial}{\partial x_k}$  for some  $1 \leq k \leq d$ , then for any  $N \gg 1$ , we have*

$$\|u_N^{(m)}\|_{L^\infty([0,1];H^s)} \rightarrow \infty$$

as  $N \rightarrow \infty$ .

*Proof.* We only prove for the case  $\partial = |\nabla|$  since the proof for the case  $\frac{\partial}{\partial x_k}$  is same. Let

$$\mathcal{A} := \{(\pm_1, \dots, \pm_m) : \pm_j \in \{+, -\} (j = 1, \dots, m)\}.$$

Since  $\mathcal{A}$  consists of  $2^m$  elements, we write

$$\mathcal{A} = \bigcup_{\alpha}^{2^m} \{\pm^{(\alpha)}\},$$

where  $\pm^{(\alpha)}$  is a  $m$ -ple of signs  $+$  and  $-$ . We denote by  $\pm_j^{(\alpha)}$  the  $j$ -th component of  $\pm^{(\alpha)}$ . A simple calculation shows that

$$\widehat{u_N^{(m)}}(t, \xi) = |\xi| \sum_{\alpha=0}^{2^m} e^{it|\xi|^4} \int_{\xi=\sum_{j=1}^m \pm_j^{(\alpha)} \xi_j} \int_0^t e^{it'(-|\xi|^4 + \sum_{j=1}^m \pm_j^{(\alpha)} |\xi_j|^4)} dt' \prod_{j=1}^m \widehat{g_N}(\xi_j).$$

From

$$\left| -|\xi|^4 + \sum_{j=1}^m \pm_j^{(\alpha)} |\xi_j|^4 \right| \lesssim N^4$$

for  $|\xi_j| \leq N$  ( $j = 1, \dots, m$ ), we have

$$|\widehat{u_N^{(m)}}(t, \xi)| \gtrsim |\xi| N^{-4} N^{-m(s+d/2)} N^{(m-1)d} \mathbf{1}_{[-N, N]^d}(\xi) \gtrsim N^{-3} N^{-m(s+d/2)} N^{(m-1)d} \mathbf{1}_{[N/2, N]^d}(\xi)$$

provided that  $t \sim N^{-4}$ . Accordingly, we obtain

$$\|u_N^{(m)}(N^{-4})\|_{H^s} \gtrsim N^{-3} N^{-m(s+d/2)} N^{(m-1)d} N^{s+d/2} \sim N^{-(m-1)s + (m-1)d/2 - 3},$$

which conclude that  $\limsup_{t \rightarrow 0} \|u_N^{(m)}(t)\|_{H^s} = \infty$  if  $s < s_c$ .  $\square$

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